

Scattering of a Klein-Gordon particle by a Hulthén potential

Jian You Guo,^{1,*} Xiang Zheng Fang,¹ and Chuan Mei Xie¹

¹*School of physics and material science, Anhui university, Hefei 230039, P.R. China*

The Klein-Gordon equation in the presence of a spatially one-dimensional Hulthén potential is solved exactly and the scattering solutions are obtained in terms of hypergeometric functions. The transmission coefficient is derived by the matching conditions on the wavefunctions and the condition for the existence of transmission resonances are investigated. It is shown how the zero-reflection condition depends on the shape of the potential.

PACS numbers: 03.65.Nk, 03.65.Pm

The study of low-momentum scattering in the Schrödinger equation in one-dimensional even potentials shows that, as momentum goes to zero, the reflection coefficient goes to unity unless the potential $V(x)$ supports a zero-energy resonance[1]. In this case the transmission coefficient goes to unity, becoming a transmission resonance[2]. Recently, this result has been generalized to the Dirac equation[3], showing that transmission resonances at $k = 0$ in the Dirac equation take place for a potential barrier $V = V(x)$ when the corresponding potential well $V = -V(x)$ supports a supercritical state. This conclusion is demonstrated in both special examples as square potential and Gaussian potential, where the phenomenon of transmission resonance is exhibited clearly in Dirac spinors in the appropriate shapes and strengths of the potentials. Except for the both special examples, the transmission resonance is also investigated in the realistic physical system. In Ref.[4], a key potential in nuclear physics is introduced, and the scattering and bound states are obtained by solving the Dirac equation in the presence of Woods-Saxon potential, which has been extensively discussed in the literature[5, 6, 7, 8, 9]. The transmission resonance is shown appearing at the spinor wave solutions with a functional dependence on the shape and strength of the potential. The presence of transmission resonance in relativistic scalar wave equations in the potential is also investigated by solving the one-dimensional Klein-Gordon equation. The phenomenon of resonance appearing in Dirac equation is reproduced at the one-dimensional scalar wave solutions with a functional dependence on the shape and strength of the potential similar to those obtained for the Dirac equation[10].

Due to the transmission resonance appearing in the realistic physical system for not only Dirac particle but also Klein Gordon particle as illustrated in the Woods-Saxon potential, it is indispensable to check the existence of the phenomenon in some other fields. Considering that the Hulthén potential[11] is an important realistic model, it has been widely used in a number of areas such as nuclear and particle physics, atomic physics, condensed matter and chemical physics[12, 13, 14, 15]. Hence, to discuss the scattering problem for a relativistic particle moving in the potential is significant, which may provide more knowledge on the transmission resonance. Recently, there have been a great deal of works to be put to the Hulthén potential in order to obtain the bound and scattering solutions in the case of relativity and non relativity[16]. However, the transmission resonance is not still checked for particle moving in the potential in the relativistic case. In this paper, we will derive the scattering solution of the Klein-Gordon equation in the presence of the general Hulthén potential, and show the phenomenon of transmission resonance as well as its relation to the parameters of the potential.

Following Ref.[10], one-dimensional Klein-Gordon equation, minimally coupled to a vector potential A^μ , is written as

$$\eta^{\alpha\beta} (\partial_\alpha + ieA_\alpha) (\partial_\beta + ieA_\beta) \phi + \phi = 0, \quad (1)$$

where the metric $\eta^{\alpha\beta} = \text{diag}(1, -1)$. For simplicity, the natural units $\hbar = c = m = 1$ are adopted, and Eq.(1) is simplified into the following form

$$\frac{d^2\phi(x)}{dx^2} + \{[E - V(x)]^2 - 1\} \phi(x) = 0. \quad (2)$$

In Eq.(2), $V(x)$ is chosen as the general Hulthén potential with the definition[11, 17] as

$$V(x) = \Theta(-x) \frac{V_0}{e^{-ax} - q} + \Theta(x) \frac{V_0}{e^{ax} - q}, \quad (3)$$

*Electronic address: jianyou@ahu.edu.cn

where all the parameters V_0 , a , and q are real and positive. To remove off the divergence of Hulthén potential, $q < 1$ is required. If $q = -1$ is taken, the Hulthén potential turns into a Woods-Saxon potential. $\Theta(x)$ is the Heaviside step function. The form of the Hulthén potential is shown in Fig.1 and 2 at different values of parameters.

From Fig.1 and 2 one readily notices that for a given value of the potential strength parameter V_0 , as q increases, the height of potential barrier increases. When $q \rightarrow 1$, the height of potential barrier goes to infinity. Similarly, the potential becomes more diffusible with the decreasing of the diffuseness parameter a .

In order to obtain the scattering solutions for $x < 0$ with $E^2 > 1$, we solve the differential equation

$$\frac{d^2\phi(x)}{dx^2} + \left\{ \left[E - \frac{V_0}{e^{-ax} - q} \right]^2 - 1 \right\} \phi(x) = 0. \quad (4)$$

On making the substitution $y = qe^{ax}$, Eq.(4) becomes

$$a^2 y^2 \frac{d^2\phi}{dy^2} + a^2 y \frac{d\phi}{dy} + \left[\left(E - \frac{V_0}{q} \frac{y}{1-y} \right)^2 - 1 \right] \phi(x) = 0. \quad (5)$$

In order to derive the solution of Eq.(5), we put $\phi = y^\mu (1-y)^\lambda f$, then Eq.(5) reduces to the hypergeometric equation

$$y(1-y)f'' + [1 + 2\mu - (2\mu + 2\lambda + 1)y]f' - \left(\lambda(1 + 2\mu) + \frac{2EV_0}{a^2 q} \right) f = 0, \quad (6)$$

where the primes denote derivatives with respect to y and the parameters μ , k , λ_\pm , ν are

$$\begin{aligned} \mu &= ik/a, \quad k = \sqrt{E^2 - 1}, \\ \lambda_\pm &= \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - (2V_0/aq)^2}, \\ \nu &= \sqrt{\mu^2 + \lambda^2 - \lambda - \frac{2EV_0}{a^2 q}}. \end{aligned}$$

The general solution of Eq.(6) can be expressed in terms of hypergeometric function as

$$\begin{aligned} f(y) &= A F(\mu - \nu + \lambda, \mu + \nu + \lambda, 1 + 2\mu; y) \\ &+ B y^{-2\mu} F(-\mu - \nu + \lambda, -\mu + \nu + \lambda, 1 - 2\mu; y). \end{aligned} \quad (7)$$

So

$$\begin{aligned} \phi_L(y) &= A y^\mu (1-y)^\lambda F(\mu - \nu + \lambda, \mu + \nu + \lambda, 1 + 2\mu; y) \\ &+ B y^{-\mu} (1-y)^\lambda F(-\mu - \nu + \lambda, -\mu + \nu + \lambda, 1 - 2\mu; y). \end{aligned} \quad (8)$$

As $x \rightarrow -\infty$, there is $y \rightarrow 0$. So, the asymptotic behavior of $\phi_L(y)$ can be written as

$$\lim_{x \rightarrow -\infty} \phi_L(y) = A q^{ik/a} e^{ikx} + B q^{-ik/a} e^{-ikx}. \quad (9)$$

Next, we consider the solution of Eq.(2) for $x > 0$. With the potential represented in Eq.(3), the differential equation to solve becomes

$$\frac{d^2\phi(x)}{dx^2} + \left\{ \left[E - \frac{V_0}{e^{ax} - q} \right]^2 - 1 \right\} \phi(x) = 0. \quad (10)$$

The analysis of the solution can be simplified making the substitution $z = qe^{-ax}$. Eq.(10) can be written as

$$a^2 z^2 \frac{d^2\phi}{dz^2} + a^2 z \frac{d\phi}{dz} + \left\{ \left[E - \frac{V_0 z}{q(1-z)} \right]^2 - 1 \right\} \phi(x) = 0. \quad (11)$$

Put $\phi = z^\mu (1 - z)^\lambda g$, Eq.(11) reduces to the hypergeometric equation

$$z(1-z)g'' + [1 + 2\mu - (2\mu + 2\lambda + 1)z]g' - \left[\lambda(1 + 2\mu) + \frac{2EV_0}{a^2q} \right]g = 0, \quad (12)$$

where the primes denote derivatives with respect to z . The general solution of Eq.(12) is

$$g(z) = C(\mu - \nu + \lambda, \mu + \nu + \lambda, 1 + 2\mu; z) + Dz^{-2\mu}F(-\mu - \nu + \lambda, -\mu + \nu + \lambda, 1 - 2\mu; z). \quad (13)$$

So,

$$\phi_R(z) = Cz^\mu(1-z)^\lambda F(\mu - \nu + \lambda, \mu + \nu + \lambda, 1 + 2\mu; z) + Dz^{-\mu}(1-z)^\lambda F(-\mu - \nu + \lambda, -\mu + \nu + \lambda, 1 - 2\mu; z). \quad (14)$$

Keeping only the solution for the transmitted wave, $C = 0$ in Eq.(14). As $x \rightarrow +\infty$ ($z \rightarrow 0$), there is

$$\phi_R(x) \rightarrow Dq^{-ik/a}e^{ikx}. \quad (15)$$

The electrical current density for the one-dimensional Klein-Gordon equation is given by the expression

$$J = \frac{i\hbar}{2m} \left(\phi \frac{d\phi^*}{dx} - \phi^* \frac{d\phi}{dx} \right). \quad (16)$$

The current as $x \rightarrow -\infty$ can be decomposed as $j_L = j_{\text{in}} - j_{\text{refl}}$ where j_{in} is the incident current and j_{refl} is the reflected one. Analogously we have that, on the right side, as $x \rightarrow \infty$ the current is $j_R = j_{\text{trans}}$, where j_{trans} is the transmitted current. Using the reflected j_{refl} and transmitted j_{trans} currents, we have that the reflection coefficient R , and the transmission coefficient T can be expressed in terms of the coefficients A , B , and D as

$$R = \frac{j_{\text{refl}}}{j_{\text{in}}} = \frac{|B|^2}{|A|^2}, \quad (17)$$

$$T = \frac{j_{\text{trans}}}{j_{\text{in}}} = \frac{|D|^2}{|A|^2}. \quad (18)$$

Obviously, R and T are not independent; they are related via the unitarity condition

$$R + T = 1. \quad (19)$$

In order to obtain R and T we proceed to equate at $x = 0$ the right ϕ_R and left ϕ_L wave functions and their first derivatives. From the matching condition we derive a system of equations governing the dependence of coefficients A and B on D that can be solved numerically.

The calculated transmission coefficient T varying with the energy E is displayed in Figs.3-6 at the different values of the parameters in the Hulthén potential. From Figs.3-6, one can see that the transmission resonance appears in all the Hulthén potential considered here. But the intensity and width of resonance as well as the condition for the existence of resonance are different, and they depend on the shape of the potential. Compared Fig.3 with Fig.4, it can be seen that the width of resonance decreases as the decreasing of diffuseness a , which is similar to that of Woods-Saxon potential as shown in Figs.3 and 5 in Ref.[10]. The same dependence can also be observed from Figs.5 and 6. Compared Fig.3 with Fig.5, one can find that the condition for the existence of transmission resonance does also relate to the parameter q . As q decreases, the height of potential barrier increases, the widths of the transmission resonance increases. The conclusion can also be obtained by comparing Fig.4 with Fig.6. In order to obtain more knowledge on the dependence of transmission resonance on the shapes of the potential, the transmission coefficient T varying with the strength of potential V_0 is plotted in Figs.7 and 8. Beside of the phenomenon of transmission resonance, similar to the Fig.3 and 4, the width of resonance decreasing as the decreasing of diffuseness a is disclosed. All these show the transmission resonances in Hulthén potential for Klein-Gordon particle possess the same rich structure with that we observe in Woods-Saxon potential.

Acknowledgments

This work was partly supported by the National Natural Science Foundation of China under Grant No. 10475001 and 10675001, the Program for New Century Excellent Talents in University of China under Grant No. NCET-05-0558, the Program for Excellent Talents in Anhui Province University, and the Education Committee Foundation of Anhui Province under Grant No. 2006KJ259B

-
- [1] R. Newton, *Scattering Theory of Waves and Particles* (Springer-Verlag, Berlin, 1982).
 - [2] D. Bohm, *Quantum Mechanics* (Prentice-Hall, Englewood Cliffs, NJ, 1951).
 - [3] N. Dombey, P. Kennedy, and A. Calogeracos, Phys. Rev. Lett. 85, 1787 (2000).
 - [4] P. Kennedy, J. Phys. A 35, 689 (2002).
 - [5] J. Y. Guo, X. Z. Fang, and F. X. Xu, Phys. Rev. A 66, 062105 (2002).
 - [6] V. Petrillo and D. Janner, Phys. Rev. A 67, 012110 (2003).
 - [7] G. Chen, Phys. Scr. 69, 257 (2004).
 - [8] J. Y. Guo, J. Meng, and F. X. Xu, Chin. Phys. Lett. 20, 602 (2003).
 - [9] A. D. Alhaidari, Phys. Rev. Lett. 87, 210405 (2001); 88, 189901(E) (2002).
 - [10] C. Rojas and V. M. Villalba, Phys. Rev. A 71, 052101 (2005).
 - [11] L. Hulthén, Ark. Mat. Astron. Fys. A 28, 5(1942).
 - [12] Y. P. Varshni, Phys. Rev. A 41, 4682(1990).
 - [13] M. Jameelt, J. Phys. A: Math. Gen. 19, 1967(1986).
 - [14] R. Barnana and R. Rajkumar R, J. Phys. A: Math. Gen. 20, 3051(1987).
 - [15] L. H. Richard, J. Phys. A: Math. Gen. 25, 1373(1992).
 - [16] C.-Y. Chen, D.-S. Sun, F.-L. Lu, Phys. Lett. A, (2007)(in press), and the References there.
 - [17] W.-C. Qiang, R.-S. Zhou, Y. Gao, Phys. Lett. A, (2007)(in press).

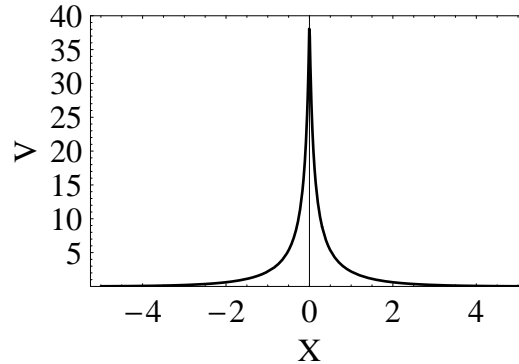


FIG. 1: Hulthén potential for $a=1.0$ and $q=0.9$ with $V_0=4$, of which the peak of barrier reaches 40.0.

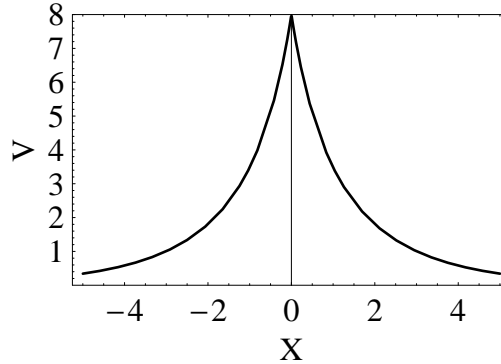


FIG. 2: Hulthén potential for $a=0.5$ and $q=0.5$ with $V_0=4$, of which the peak of barrier reaches 8.0.

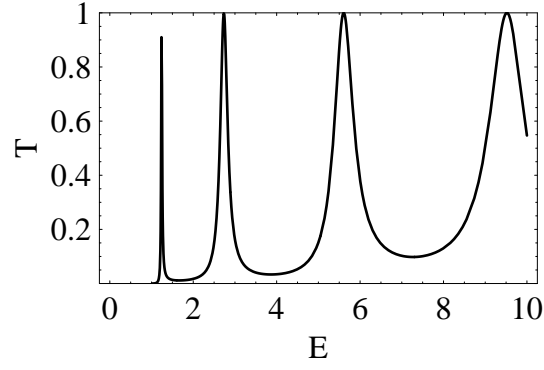


FIG. 3: The transmission coefficient for the relativistic Hulthén potential barrier. The plot illustrate T for varying energy E with $V_0 = 4$, $a = 1$, and $q = 0.9$.

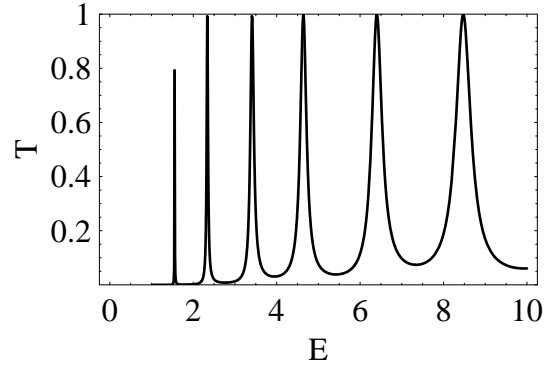


FIG. 4: Similar to Fig.3, but with $V_0 = 4$, $a = 0.5$, and $q = 0.9$.

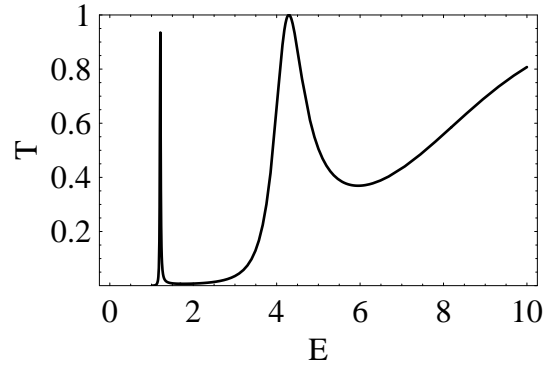


FIG. 5: Similar to Fig.3, but with $V_0 = 4$, $a = 1$, and $q = 0.5$.

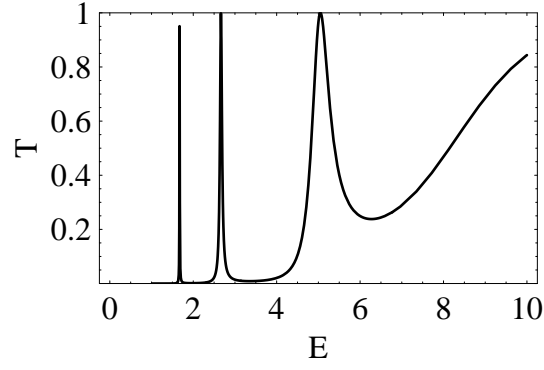


FIG. 6: Similar to Fig.3, but with $V_0 = 4$, $a = 0.5$, and $q = 0.5$.

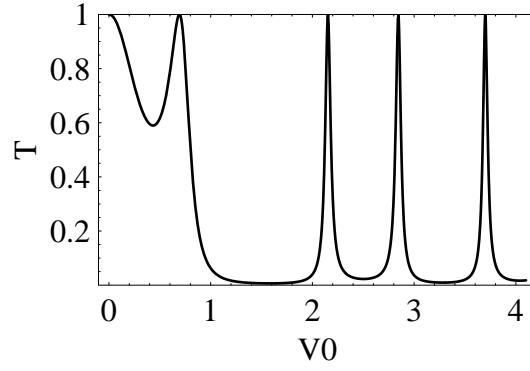


FIG. 7: The transmission coefficient for the relativistic Hulthén potential barrier. The plot illustrate T for varying barrier height V_0 with $E = 2$, $a = 1$, and $q = 0.9$.

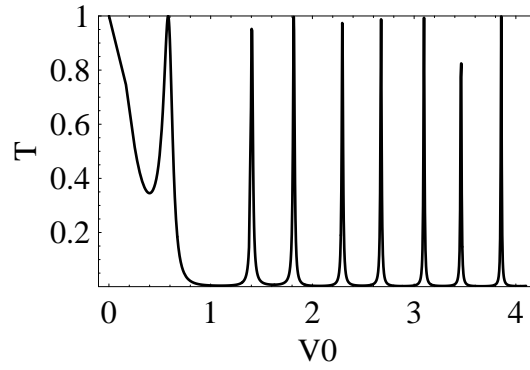


FIG. 8: Similar to Fig.7, but with $E = 2$, $a = 0.5$, and $q = 0.9$.